Weakening the Legendre Conjecture

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Abstract

The world of primes has many gaps between evidence and theorems. Here, we review Legendre's conjecture on primes between consecutive squares and recent progress on the weaker question of primes between consecutive larger powers. Assuming the Riemann hypothesis (RH), we observe that a recent result of Emanuel Carneiro, Micah Milinovich and Kannan Soundararajan, combined with a large-scale computation by Jonathan Sorenson and Jonathan Webster, implies the existence of primes between $x^{2+\delta}$ and $(x+1)^{2+\delta}$ for all real $x \geq 1$ when $\delta \geq 1/4$. For smaller values of $\delta > 0$, we provide an explicit bound $x_0 = x_0(\delta)$ such that primes exist in these intervals whenever $x \geq x_0$ (again assuming RH). We conclude with an application to Mills-type prime-generating constants.

1 Introduction

Adrien-Marie Legendre famously conjectured that there is always a prime between consecutive squares. In 1912, this conjecture was featured by Edmund Landau at the International Congress of Mathematicians as one of four "unattackable" problems in number theory (the other three problems being the Goldbach conjecture, the twin prime conjecture, and the conjecture that there are infinitely many primes of the form n^2+1).

Conjecture 1.1 (Legendre)**.** *There is a prime between n^2 and $(n+1)^2$ for all positive integers n.*

This conjecture is usually stated for positive integers n but it appears to also hold when n is replaced by a positive, real number. In other words, conjecturally, every interval $[x^2, (x+1)^2]$, where $x \geq 1$ is a real number, contains at least one prime. Throughout, we will focus on this slight strengthening of Legendre's conjecture.

While Legendre's conjecture has withstood proof for more than a century since Landau's endorsement, it is natural to wonder what can be said about weaker versions. For instance, it is known that there is a prime between n^3 and $(n+1)^3$ for all sufficiently large n. In fact, this is known to be true for $n \geq \exp(\exp(32.537))$ due to recent work of Michaela Cully-Hugill [CH23a, Theorem 4.3], improving a similar earlier bound of Adrian Dudek

[Dud16]. Stronger results are available if we assume that the Riemann hypothesis (RH) is true (this is because, as is recalled at the beginning of Section 2, RH is equivalent to certain estimates for the number of primes less than some x). A goal of this article is to illustrate this explicitly in Theorem 1.2 below, which follows from combining a recent result of Emanuel Carneiro, Micah Milinovich and Kannan Soundararajan [CMS19] with a large-scale computation by Jonathan Sorenson and Jonathan Webster [SW25].

Theorem 1.2. *Suppose the Riemann hypothesis is true, and let $\delta > 0$.*

(a) If $\delta \geq 1/4$, then the interval $[x^{2+\delta}, (x+1)^{2+\delta}]$ contains a prime for all $x \geq 1$.

(b) If $\delta < 1/4$, then the interval $[x^{2+\delta}, (x+1)^{2+\delta}]$ contains a prime for all

$$x \geq [\beta(\log(\beta) + \log(\log(\beta)))]^{\beta}, \quad \beta = 2/\delta.$$

With more work, this result can be sharpened. For instance, as indicated at the end of Section 3, the bound for δ in part (a) can be improved to 0.2253. On the other hand, obtaining a result like $\delta \geq 1/5$ in part (a) seems out of reach with the current state of computation and theory.

Such Legendre-type results have an appealing application to prime-generating constants originating with William Mills [Mil47], who famously proved that there exists a real number A such that $\lfloor A^{3^n} \rfloor$ is a prime for every positive integer n. Mills' proof is based on a result of Albert Ingham [Ing37] from which he deduces that there is always a prime between n^3 and $(n+1)^3$ for all sufficiently large n. By the special case $\delta = 1$ of Theorem 1.2, which was previously proved by Chris Caldwell and Yuanyou Cheng [CC05], this is in fact true for all positive integers n if we are willing to assume the Riemann hypothesis. Caldwell and Cheng use that fact to show that, assuming RH, the number

$$A = 1.306377883863080690468614492602\ldots \tag{1}$$

is the smallest number with the property that $\lfloor A^{3^n} \rfloor$ is a prime for all n. Without assuming RH, Christian Elsholtz [Els20] has produced unconditional variants that generate only primes. Namely, to high precision, Elsholtz unconditionally gives constants B and C such that $\left\lfloor B^{10^{10n}} \right\rfloor$ and $\left\lfloor C^{3^{13n}} \right\rfloor$ are prime for all n. In Section 5, assuming RH, we show that (a minor strengthening of) Theorem 1.2 allows us to determine, for any fixed $\alpha \geq 2.25$, the smallest constant A such that $\lfloor A^{\alpha^n} \rfloor$ is a prime for all n.

In Section 2, we begin by demonstrating how the Riemann hypothesis readily implies that there is a prime between $x^{2+\delta}$ and $(x+1)^{2+\delta}$ for all sufficiently large x. The explicit bounds provided by Theorem 1.2 on x are then discussed in Section 3. In Section 4, we briefly consider the question of reducing the exponents of x and $x+1$ to values less than 2, in which case one can again ask whether there are primes between such smaller powers. In that case, it is truly necessary to restrict to values of x that are sufficiently large. The

resulting conjecture, however, remains open even assuming the Riemann hypothesis. We conclude in Section 5 with an application to prime-generating constants originating with Mills [Mil47].

In the remainder of this introduction, we compare Theorem 1.2 with various existing results that do not rely on the Riemann hypothesis. For instance, it is known that Theorem 1.2 holds for large enough values of $\alpha = 2 + \delta$ unconditionally. Specifically, Dudek [Dud16] has proved, without relying on RH, that if $\alpha \geq 5 \cdot 10^9$, then there is a prime between n^α and $(n+1)^\alpha$ for all $n \geq 1$. Caitlin Mattner [Mat17] subsequently improved this result to hold for all $\alpha \geq 1{,}438{,}989$. Very recently, Michaela Cully-Hugill and Daniel Johnston [CHJ25] have unconditionally proved that, for $\alpha = 90$, there is always a prime between n^α and $(n+1)^\alpha$ for all $n \geq 1$. This improves earlier results [CH23b] and [CHJ23] that showed the same conclusion for $\alpha = 155$ and $\alpha = 140$.

Let p_n denote the nth prime. Numerous results can be found in the literature that prove, for fixed θ,

$$p_{n+1} - p_n < p_n^\theta \tag{2}$$

for all sufficiently large n. The case $\theta = 1$ was postulated by Joseph Bertrand in 1845 and proved by Pafnuty Chebyshev in 1852 (in this case (2) holds for all $n \geq 1$). If $\theta < 1$ the inequalities (2) are equivalent to the intervals $[x - x^\theta, x]$ containing at least one prime for all sufficiently large x. The first such result was due to Guido Hoheisel [Hoh30] who showed that one can take $\theta = 32999/33000$. For a history of results of this kind, we refer to [BH96], where this is improved to $\theta = 0.535$. At present, the strongest such result is due to Roger Baker, Glyn Harman and János Pintz [BHP01] who prove that $\theta = 0.525$ is possible. Assuming the Riemann hypothesis, it is known that we can choose any $\theta > 1/2$, while a conjecture of Harald Cramér [Cra36] implies that, in fact, any $\theta > 0$ is possible. We refer to Section 4 for this conjecture on the true nature of the prime gaps $p_{n+1} - p_n$. Here, note that results of the type (2) yield weak versions of Legendre's conjecture as follows.

Proposition 1.3. *Fix $0 < \theta < 1$ and let $\alpha = \frac{1}{1-\theta}$. If the inequality (2) holds for all sufficiently large n then the intervals $[x^\alpha, (x+1)^\alpha]$ contain a prime for all sufficiently large x.*

Proof. Suppose that $[x^\alpha, (x+1)^\alpha]$ does not contain a prime, and choose n so that $p_n < x^\alpha < (x+1)^\alpha < p_{n+1}$. In that case,

$$\frac{p_{n+1} - p_n}{p_n^\theta} > \frac{(x+1)^\alpha - x^\alpha}{x^{\alpha\theta}} > \frac{\alpha x^{\alpha-1}}{x^{\alpha\theta}} = \alpha x^{\alpha(1-\theta)-1} = \alpha > 1,$$

which contradicts (2). □

For $\theta = 0.525$, we have $\frac{1}{1-\theta} \approx 2.1053$. Hence, the result by Baker, Harman and Pintz [BHP01] implies the following.

Corollary 1.4 (Baker, Harman, Pintz [BHP01]). *If $\alpha \geq 2.106$, then there exists x_0 such that the intervals $[x^\alpha, (x+1)^\alpha]$ contain a prime for all $x \geq x_0$.*

Baker, Harman and Pintz [BHP01] remark that "with enough effort" one could work out an explicit value for the lower bound x_0. However, due to the intricate techniques involved, both the required effort as well as the resulting bound would likely be enormous. On the other hand, as seen in Theorem 1.2, if we assume RH then it is not too hard to obtain explicit lower bounds for all $\alpha > 2$.

2 Weak versions of the Legendre conjecture

Let $\pi(x)$ denote the prime counting function and let $\mathrm{Li}(x)$ be the logarithmic integral

$$\mathrm{Li}(x) = \int_2^x \frac{\mathrm{d}t}{\log(t)}.$$

It is well-known that the Riemann hypothesis is equivalent to having the bounds, as $x \to \infty$,

$$\pi(x) = \mathrm{Li}(x) + O(x^{1/2+\varepsilon}) \tag{3}$$

for all fixed $\varepsilon > 0$. The exponent $1/2$ in this bound directly reflects the fact that the RH predicts all non-trivial zeros of the Riemann zeta function to have real part $1/2$. Unconditionally, it is not even known whether the exponent $1/2$ can be replaced with σ for any $\sigma < 1$ (corresponding to the fact that there is no value $\sigma < 1$ for which we can currently prove that all non-trivial zeros of the Riemann zeta function have real part less than σ). We remark that Helge von Koch [VK01] further proved that the Riemann hypothesis is equivalent to

$$\pi(x) = \mathrm{Li}(x) + O\left(\sqrt{x}\log(x)\right). \tag{4}$$

(Note that inequality (4) is clearly stronger than the bound (3) for any $\varepsilon > 0$.)

We now observe that any bound of the form (3) implies that there are primes between corresponding consecutive powers.

Lemma 2.1. *Suppose the bound (3) holds for a certain $\varepsilon \in \left(0, \frac{1}{2}\right)$ and let $\alpha > \frac{2}{1-2\varepsilon}$. Then the interval $[x^\alpha, (x+1)^\alpha]$ contains a prime for all sufficiently large x.*

Proof. Suppose that $\alpha > 2/(1-2\varepsilon)$. Since $O((x+1)^\beta) = O(x^\beta)$ as $x \to \infty$ for any $\beta > 0$, the bound (3) implies that

$$\begin{aligned} \pi((x+1)^\alpha) - \pi(x^\alpha) &= \mathrm{Li}((x+1)^\alpha) - \mathrm{Li}(x^\alpha) + O(x^{\alpha(1/2+\varepsilon)}) \\ &= \int_{x^\alpha}^{(x+1)^\alpha} \frac{\mathrm{d}t}{\log(t)} + O(x^{\alpha(1/2+\varepsilon)}). \end{aligned}$$

On the other hand,

$$\int_{x^\alpha}^{(x+1)^\alpha} \frac{\mathrm{d}t}{\log(t)} > \frac{(x+1)^\alpha - x^\alpha}{\alpha \log(x+1)} > \frac{x^{\alpha-1}}{\log(x+1)},$$

where we used $(x+1)^\alpha > x^\alpha + \alpha x^{\alpha-1}$ for the final inequality.

We now note that the inequality $\alpha - 1 > \alpha(1/2 + \varepsilon)$ is equivalent to our assumption that $\alpha > 2/(1-2\varepsilon)$. Therefore, put together, we have

$$\pi((x+1)^\alpha) - \pi(x^\alpha) > \frac{x^{\alpha-1}}{\log(x)}(1 + o(1)).$$

In particular, we have $\pi((x+1)^\alpha) - \pi(x^\alpha) > 1$ for sufficiently large x. □

If the Riemann hypothesis is true, then we can choose any $\varepsilon > 0$ in Lemma 2.1. It follows that the conclusion is true for all $\alpha > 2$ and we obtain the following.

Corollary 2.2. *Suppose the Riemann hypothesis is true, and let $\alpha > 2$ be fixed. Then the interval $[x^\alpha, (x+1)^\alpha]$ contains a prime for all sufficiently large x.*

Let us note that, using a strong result of Ingham [Ing37], the Riemann hypothesis in Corollary 2.2 can be replaced with the Lindelöf hypothesis which is known to be weaker than RH. Namely, Ingham proves that if we have

$$\zeta\left(\frac{1}{2} + it\right) = O(t^c), \quad t \to \infty,$$

for some constant $c > 0$, then the inequalities $p_{n+1} - p_n < p_n^\theta$ in (2) hold for large enough n if $\theta > \frac{1+4c}{2+4c}$. The Lindelöf hypothesis posits that we can choose c to be arbitrarily close to 0. If true, it follows that any $\theta > \frac{1}{2}$ can be chosen in (2). Thus, Proposition 1.3 implies Corollary 2.2 with the Lindelöf hypothesis in place of the Riemann hypothesis.

3 Explicit bounds under RH

The error term in (4) suggests the length of short intervals that are guaranteed to contain a prime. At present, the strongest known result is the following due to Carneiro, Milinovich and Soundararajan [CMS19].

Theorem 3.1 (Carneiro, Milinovich, Soundararajan [CMS19])**.** *Suppose the Riemann hypothesis is true. Then, for all $x \geq 4$, there is a prime in $\left[x, x + \frac{22}{25}\sqrt{x}\log(x)\right]$.*

Theorem 3.1 improves intervals established earlier by different techniques, for instance, by Olivier Ramaré and Yannick Saouter [RS03], Dudek [Dud15] as well as Dudek, Loïc Grenié and Giuseppe Molteni [DGM16]. From Theorem 3.1, we can readily conclude the

following weak version of Legendre's conjecture. We note that, given $\alpha > 2$, the inequality (5) holds for sufficiently large x (because $\alpha/2 - 1 > 0$). We are using a slightly smaller interval than $[x^\alpha, (x+1)^\alpha]$ in the conclusions because that is convenient for the application to prime-generating constants in Lemma 5.1.

Corollary 3.2. *Suppose the Riemann hypothesis is true, and let $\alpha > 2$ be fixed. If*

$$x^{\alpha/2-1} \geq \frac{22}{25}\log(x), \quad x \geq 1, \tag{5}$$

then there is a prime in the interval $[x^\alpha, (x+1)^\alpha - 1)$.

Proof. The statement is clearly true for $x = 1$. In the sequel, we therefore assume $x > 1$. Observe that the interval $[x^\alpha, (x+1)^\alpha - 1)$ has length exceeding $\alpha x^{\alpha-1} \geq 2x > 2$. Hence, since the first prime gap larger than 2 occurs between 7 and 11, we may further assume $x^\alpha > 7$. We set $y = x^\alpha$ with the intention of applying Theorem 3.1 (with y in place of x; note that, by our assumptions, the condition $y \geq 4$ is satisfied) and observe that

$$y + \frac{22}{25}\sqrt{y}\log(y) = x^\alpha + \frac{22}{25}\alpha x^{\alpha/2}\log(x) < (x+1)^\alpha - 1$$

if and only if

$$(x+1)^\alpha - 1 - x^\alpha > \frac{22}{25}\alpha x^{\alpha/2}\log(x). \tag{6}$$

Since the left-hand side satisfies $(x+1)^\alpha - 1 - x^\alpha > \alpha x^{\alpha-1}$, it is sufficient (though not necessary) for (6) that

$$x^{\alpha-1} \geq \frac{22}{25}x^{\alpha/2}\log(x),$$

which is equivalent to the inequality (5). For those x satisfying (5), it therefore follows from Theorem 3.1 that the interval $[x^\alpha, (x+1)^\alpha - 1)$ is guaranteed to contain a prime, as claimed. □

The following result provides the explicit bound (7) on x that implies the inequality (5).

Lemma 3.3. *Suppose that $2 < \alpha \leq 2\left(1 + \frac{1}{e}\right) \approx 2.736$ and define $\beta = 2/(\alpha - 2)$. Then the inequality (5) holds for all*

$$x \geq [\beta(\log(\beta) + \log(\log(\beta)))]^\beta. \tag{7}$$

Proof. The inequality (5) takes the form $x \geq (C\log(x))^\beta$ with $C = 22/25$. It follows directly from

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{x}{\log(x)^\beta} = \frac{\log(x) - \beta}{\log(x)^{\beta+1}}$$

that $x/\log(x)^\beta$ is an increasing function for $x \geq e^\beta$. On the other hand, the bounds on α imply that $\beta \geq e$. In particular, $\log(\log(\beta)) \geq 0$ so that it follows from (7) that $x \geq \beta^\beta \geq e^\beta$. It therefore suffices to show that the inequality (5) holds for

$$x = [\beta(\log(\beta) + \log(\log(\beta)))]^\beta.$$

With this value for x fixed,

$$\begin{aligned} \log(x) &= \beta \log(\beta(\log(\beta) + \log(\log(\beta)))) \\ &= \beta \left(\log(\beta) + \log(\log(\beta)) + \log\left(1 + \frac{\log(\log(\beta))}{\log(\beta)}\right)\right). \end{aligned}$$

It follows that

$$x^{1/\beta} - \frac{22}{25}\log(x) = \frac{\beta}{25}\left(3(\log(\beta) + \log(\log(\beta))) - 22\log\left(1 + \frac{\log(\log(\beta))}{\log(\beta)}\right)\right),$$

and it only remains to observe that the right-hand side is positive (at least for $\beta \geq e$). This is left as a calculus exercise (the coefficients 3 and 22 are not optimal). □

Sorenson and Webster [SW25] recently described an algorithm to efficiently test a slightly stronger version of Legendre's conjecture due to Ludvig Oppermann, namely that there are primes between n^2 and $n(n+1)$ as well as between $n(n+1)$ and $(n+1)^2$ for all integers $n > 1$. Using that algorithm they verify that this is true for all $n \leq N$ where $N = 7.05 \cdot 10^{13}$. The following observation allows us to conclude from this computation that the intervals $[x^\alpha, (x+1)^\alpha]$ contain a prime provided that $1 \leq x^{\alpha/2} < N$. In particular, letting $N \to \infty$, we see that Oppermann's conjecture implies that, for every $\alpha \geq 2$ and $x \geq 1$, there is at least one prime in the interval $[x^\alpha, (x+1)^\alpha]$.

Proposition 3.4. *Let N be an integer and suppose that the intervals $[n^2, n(n+1)]$ and $[n(n+1), (n+1)^2]$ each contain a prime for all $n \in \{1, 2, \ldots, N\}$. Then, for every $\alpha \geq 2$ and $x \geq 1$, the interval $[x^\alpha, (x+1)^\alpha]$ contains a prime provided that $x^{\alpha/2} < N$.*

Proof. Consider a specific interval $[x^\alpha, (x+1)^\alpha]$ for $x \geq 1$ and $\alpha \geq 2$. If $y = x^{\alpha/2}$ then $[y^2, (y+1)^2]$ is contained in $[x^\alpha, (x+1)^\alpha]$. It therefore suffices to show that every interval $[y^2, (y+1)^2]$ contains a prime provided that $1 \leq y < N$.

Set $m = \lfloor y \rfloor$ so that $m^2 \leq y^2 < (m+1)^2 \leq (y+1)^2$. First, let us assume that $m(m+1) \geq y^2$. In that case, we have $[m(m+1), (m+1)^2] \subseteq [y^2, (y+1)^2]$. Hence both intervals contain a prime provided that $m \leq N$. Second, we assume that $m(m+1) < y^2$. In that case,

$$(y+1)^2 = y^2 + (2y+1) > m(m+1) + (2m+1) = (m+1)(m+2) - 1.$$

Consequently, we find that $[(m+1)^2, (m+1)(m+2) - 1] \subset [y^2, (y+1)^2]$. Hence both intervals contain a prime provided that $m + 1 \leq N$. □

Equipped with Corollary 3.2 as well as Lemma 3.3 and Proposition 3.4, we complete the proof of Theorem 1.2 stated in the introduction. Since it is useful for the application in Section 5, we will actually prove Theorem 1.2 with the intervals $[x^{2+\delta}, (x+1)^{2+\delta}]$ replaced with the slightly smaller intervals $[x^{2+\delta}, (x+1)^{2+\delta} - 1)$.

Proof of Theorem 1.2. If $\delta < 1/4$, then the result follows from combining Corollary 3.2 with Lemma 3.3. In the remainder, we therefore assume $\delta \geq 1/4$. Note that if the interval $[x^\alpha, (x+1)^\alpha - 1)$ does not contain a prime for some $x \geq 1$, then, for $\beta < \alpha$, the interval $[y^\beta, (y+1)^\beta - 1)$ with $y = x^{\alpha/\beta} \geq 1$ also does not contain a prime. It therefore suffices to consider the case $\delta = 1/4$.

Let $\delta = 1/4$ and, hence, $\alpha = 2 + \delta = 9/4$. From Lemma 3.3 with $\beta = 2/\delta = 8$ applied to Corollary 3.2, we find that there is a prime in $[x^\alpha, (x+1)^\alpha - 1)$ provided that $x \geq [\beta(\log(\beta) + \log(\log(\beta)))]^\beta \approx 6.55 \cdot 10^{10}$. On the other hand, the computations by Sorenson and Webster [SW25] allow us to apply Proposition 3.4 with $N = 7.05 \cdot 10^{13}$ to conclude that all intervals $[y^2, (y+1)^2]$ contain a prime provided that $1 \leq y < N$. If $y = x^{\alpha/2}$ then $[y^2, (y+1)^2]$ is contained in $[x^\alpha, (x+1)^\alpha - 1)$ provided that x is large enough. For $\alpha = 9/4$ it is sufficient that $x \geq 2$. Since smaller values of x are readily verified directly, it follows that the intervals $[x^\alpha, (x+1)^\alpha - 1)$ contain a prime provided that $x^{\alpha/2} < N$. In the present case, this bound on x takes the form $x < N^{2/\alpha} \approx 2.04 \cdot 10^{12}$. This exceeds $6.55 \cdot 10^{10}$ and thus completes the proof. □

Clearly, this final argument is not sharp. Indeed, the same argument goes through provided that $[\beta(\log(\beta) + \log(\log(\beta)))]^\beta < N^{2/\alpha}$ or, equivalently,

$$[\beta(\log(\beta) + \log(\log(\beta)))]^{\beta+1} < N.$$

Using the value $N = 7.05 \cdot 10^{13}$ from [SW25], this allows us to lower the bound slightly for δ in Theorem 1.2 from 1/4 to 0.2275. By replacing the bound (7) with the inequality (5), we can further reduce δ to 0.2253.

4 Primes between consecutive smaller powers

Inspired by Legendre's conjecture, one may wonder whether there are primes between consecutive smaller powers. More precisely, given $\alpha > 1$, what can we say about primes between x^α and $(x+1)^\alpha$? Here, we take $\alpha > 1$ since if $\alpha = 1$ then it is trivially the case that there are arbitrarily large x such that the intervals $[x^\alpha, (x+1)^\alpha]$ contain no prime.

Example 4.1. A quick numerical search suggests that, for integers n, there is always a prime between $n^{3/2}$ and $(n+1)^{3/2}$ provided that $n > 1051$. Indeed, it appears that the interval $[n^{3/2}, (n+1)^{3/2}]$ does not contain a prime precisely if n is one of 10, 20, 24, 27, 32, 65, 121, 139, 141, 187, 306, 321, 348, 1006, 1051. This is sequence A144140 on the OEIS [OEI26] where we find it conjectured that our computed list of exceptional values is complete.

In general, the following is expected to be true.

Conjecture 4.2. *Let $\alpha > 1$ be fixed. Then there is a prime between x^α and $(x+1)^\alpha$ for all sufficiently large x.*

This conjecture is another way to state that it is expected that $p_{n+1} - p_n = O(p_n^\delta)$ for any $\delta > 0$. While this is open, even assuming the Riemann hypothesis, stronger conjectures have been put forth. Specifically, Cramér [Cra36] conjectured in 1936 that gaps between consecutive primes are remarkably small in the following sense.

Conjecture 4.3 (Cramér [Cra36])**.** *As $n \to \infty$, we have $p_{n+1} - p_n = O(\log^2(p_n))$.*

Due to the extensive computations by Tomás Oliveira e Silva, Siegfried Herzog and SilvioPardi [OeSHP14, Section 2.2.1] in the context of verifying the Goldbach conjecture, we know that $p_{n+1} - p_n < \log^2(p_n)$ for all primes $11 \le p_n \le 4 \cdot 10^{18}$. On the other hand, Legendre's conjecture implies that $p_{n+1} - p_n = O\left(\sqrt{p_n}\right)$ while, assuming the Riemann hypothesis, Cramér [Cra36] proved that $p_{n+1} - p_n = O\left(\sqrt{p_n}\log(p_n)\right)$.

5 Prime-generating constants

The constructions of Mills-type prime-generating constants in [Mil47], [CC05] and [Els20], mentioned in the introduction, are based on the following lemma which connects these with Legendre-type results about the existence of primes between consecutive powers.

Lemma 5.1. *Let $\alpha > 1$. Suppose that the intervals $[n^\alpha, (n+1)^\alpha - 1)$ contain a prime for all $n \ge n_0$. Then there exists a real number A such that $\lfloor A^{\alpha^n} \rfloor$ is a prime for every positive integer n.*

The general construction underlying this lemma is essentially given by Ivan Niven [Niv51]. Since it is rather simple and instrumental for the subsequent examples, we include a proof below. We first observe that the intervals $[n^\alpha, (n+1)^\alpha - 1)$ in the lemma are equivalent to the intervals $[n^\alpha, (n+1)^\alpha]$ if $\alpha \ge 2$ is an integer (since in that case neither $(n+1)^\alpha$ nor $(n+1)^\alpha - 1$ are prime). On the other hand, we note that, with the present application in mind, our proof of Theorem 1.2 actually shows that, assuming the RH, the slightly smaller intervals $[x^\alpha, (x+1)^\alpha - 1)$ contain a prime for all $x \ge 1$ provided that $\alpha \ge 2.25$. For $\alpha > 2$ these intervals contain a prime for all sufficiently large x, and this conclusion continues to hold for all $\alpha > 1$ if Conjecture 4.2 is true.

Proof of Lemma 5.1. Pick q_1 to be any prime larger than n_0. By assumption, we can then select primes $q_2, q_3, \ldots$ in such a way that $q_{n+1} \in [q_n^\alpha, (q_n+1)^\alpha - 1)$. Note that we have

$$q_n^\alpha \le q_{n+1} < q_{n+1} + 1 < (q_n+1)^\alpha$$

and, hence,

$$q_n^{\alpha^{-n}} \le q_{n+1}^{\alpha^{-(n+1)}} < (q_{n+1}+1)^{\alpha^{-(n+1)}} < (q_n+1)^{\alpha^{-n}}. \tag{8}$$

It follows that the sequences $q_n^{\alpha^{-n}}$ and $(q_n+1)^{\alpha^{-n}}$ are weakly increasing and decreasing, respectively. Being bounded by the terms of the latter, the sequence $q_n^{\alpha^{-n}}$ has a finite limit A as $n \to \infty$. It follows from (8) that

$$q_n \le A^{\alpha^n} < q_n + 1.$$

This is equivalent to $\lfloor A^{\alpha^n} \rfloor = q_n$. □

Note that the inequalities (8) imply that

$$q_n^{\alpha^{-n}} \le A < (q_n+1)^{\alpha^{-n}} \tag{9}$$

and that this allows us to estimate the constant A produced by the proof of Lemma 5.1 to high precision.

Example 5.2. Assuming the RH, the case $\delta = 1$ of Theorem 1.2 implies that the intervals $[n^3, (n+1)^3]$ and, hence, $[n^3, (n+1)^3 - 1)$ contain a prime for all integers $n \ge 1$. In the proof of Lemma 5.1 we can therefore choose $q_1 = 2$ and then inductively select q_{n+1} to be the least prime exceeding q_n^3. Accordingly,

$$q_2 = 11, q_3 = 1361, q_4 = 2521008887, \ldots$$

and we can estimate the resulting constant A using (9). For $n = 4$, the two sides of (9) differ by less than $6.4 \cdot 10^{-12}$, thus producing the value of $A = 1.30637788\ldots$ in (1) with 11 correct decimal digits. Using $n = 10$, Caldwell and Cheng [CC05] computed A to more than 6850 digits. Since we made the minimal possible choices for each of the primes q_n, it is not hard to see that this value A is the smallest real number such that $\lfloor A^{3^n} \rfloor$ is a prime for all n.

We can proceed likewise for any exponent $\alpha = 2 + \delta$ where $\delta \ge 1/4$. Indeed, as indicated above, our proof of Theorem 1.2 actually shows that, assuming the RH, the intervals $[x^\alpha, (x+1)^\alpha - 1)$ contain a prime for all $x \ge 1$. In the proof of Lemma 5.1 we can therefore again choose $q_1 = 2$ and then inductively select q_{n+1} to be the least prime exceeding q_n^α. For instance, for $\alpha = 2.25$ this results in

$$q_2 = 5, q_3 = 41, q_4 = 4259, q_5 = 146535287, q_6 = 2362490078520203903, \ldots$$

as well as the corresponding value

$$A = 1.385510850556131889057735009841\ldots$$

for the smallest real number, assuming the RH, such that $\lfloor A^{2.25^n} \rfloor$ is a prime for all n.

By Lemma 5.1, Legendre's conjecture implies that there is a real number A such that $\lfloor A^{2^n} \rfloor$ is a prime for all positive integers n. Remarkably, Kaisa Matomäki [Mat10] was able to prove the existence of such a number A without assuming Legendre's conjecture. Her construction is different and more intricate than the one in Lemma 5.1, and does not provide a specific value for A.

Example 5.3. Assuming the Legendre conjecture, we can proceed as in Example 5.2 for $\alpha = 2$ in place of 3 to construct and approximate the smallest A such that $\lfloor A^{2^n} \rfloor$ is a prime for all n. That is, we let $q_1 = 2$ and then inductively define q_{n+1} to be the least prime exceeding q_n^2. This results in

$$q_2 = 5, q_3 = 29, q_4 = 853, q_5 = 727613, q_6 = 529420677791, \ldots$$

as well as the corresponding value

$$A = 1.524699960538094359923363575688\ldots.$$

By construction, this A has the property that $\lfloor A^{2^n} \rfloor$ is a prime for all n if we can prove the following.

Conjecture 5.4. *Define the sequence $q_1, q_2, q_3, \ldots$ by $q_1 = 2$ and by letting q_{n+1} be the least prime exceeding q_n^2. Then $q_{n+1} < (q_n + 1)^2$ for all n.*

Although Conjecture 5.4 (or a variation with the initial prime $q_1 = 2$ replaced by another fixed prime) is weaker than Legendre's conjecture, it is likely of a similar difficulty and thus still out of reach.

6 Relaxing Legendre's conjecture in other directions

With a particular interest in the case $\alpha = 2$, we have discussed the question of whether there is always a prime between x^α and $(x+1)^\alpha$, possibly assuming that x is large enough. For $\alpha > 2$, there are weaker versions of Legendre's conjecture and Theorem 1.2 indicates what is currently known, assuming the Riemann hypothesis.

Legendre's conjecture can be fruitfully weakened in other directions as well. For instance, Danilo Bazzanella [Baz00] has proven that Legendre's conjecture holds for most intervals $[n^2, (n+1)^2]$ in the sense that, for every $\varepsilon > 0$, each interval $[n^2, (n+1)^2] \subseteq [1, N]$, with at most $O(N^{1/4+\varepsilon})$ exceptions, contains a prime (and, in fact, the expected number of primes). Further stronger bounds on the number of possible exceptions are provided in [Baz00] and [Baz11] under the Riemann hypothesis as well as another weaker heuristic hypothesis. In a similar direction, it is known that even smaller intervals "almost always" contain a prime. For instance, a result of Chaohua Jia [Jia96] implies that the intervals $[n, n + n^\theta]$ with $\theta = \frac{1}{20} + \varepsilon$ contain a prime number for all $n \in [N, 2N]$ with at most $o(N)$ exceptions. By a recent result of Larry Guth and James Maynard [GM26] (Corollary 1.4)

such intervals with $\theta = \frac{2}{15} + \varepsilon$ almost always contain the expected number of primes. We refer the interested reader to these papers for more details and history.

Approximating the Legendre conjecture in a different direction, Jing-Run Chen [Che75] showed that there are integers with at most two prime factors between sufficiently large consecutive squares. Recently, Dudek and Johnston [DJ26] showed that the restriction to sufficiently large squares can be removed if one allows integers with at most four prime factors.

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