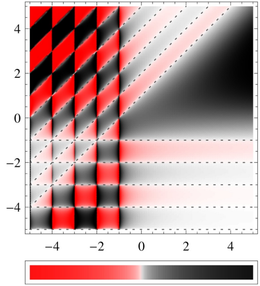
Gaussian binomial coefficients with negative arguments

Colloquium
University of South Alabama

Armin Straub

September 29, 2022

University of South Alabama



includes joint work with:

Sam Formichella
(University of South Alabama)

Basic q-analogs

IDEA A q-analog reduces to the classical object in the limit $q \to 1$.

DEF

- q-number: $$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + \ldots q^{n-1}$$

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D1

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$$\binom{6}{2} = \frac{6 \cdot 5}{2 \cdot 1} = 3 \cdot 5$$

$$\binom{6}{2}_q = \frac{(1 + q + q^2 + q^3 + q^4 + q^5) \cdot (1 + q + q^2 + q^3 + q^4)}{(1 + q) \cdot 1}$$

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$$= \underbrace{(1 - q + q^2)}_{= \Phi_6(q)} \underbrace{(1 + q + q^2)}_{= [3]_q} \underbrace{(1 + q + q^2 + q^3 + q^4)}_{= [5]_q}$$

$\Phi_6(1) = 1$ becomes invisible

The primes of the q-world

DEF The nth cyclotomic polynomial:

$$\Phi_n(q) = \prod_{\substack{1 \leqslant k < n \\ (k,n)=1}} (q - \zeta^k) \qquad \text{where } \zeta = e^{2\pi i/n}$$

irreducible polynomial (nontrivial; Gauss!) with **integer** coefficients

- $[n]_q = \dfrac{q^n - 1}{q - 1} = \prod_{\substack{1 < d \leqslant n \\ d|n}} \Phi_d(q)$ For primes: $[p]_q = \Phi_p(q)$

EG

$$\Phi_2(q) = q + 1, \quad \Phi_3(q) = q^2 + q + 1, \quad \Phi_4(q) = q^2 + 1$$

$$\Phi_5(q) = q^4 + q^3 + q^2 + q + 1$$

$$\Phi_6(q) = q^2 - q + 1, \quad \Phi_9(q) = q^6 + q^3 + 1$$

$$\vdots$$

$$\begin{aligned}\Phi_{102}(q) = {} & q^{32} + q^{31} - q^{29} - q^{28} + q^{26} + q^{25} - q^{23} - q^{22} + q^{20} \\ & + q^{19} - q^{17} - q^{16} - q^{15} + q^{13} + q^{12} - q^{10} - q^9 + q^7 \\ & + q^6 - q^4 - q^3 + q + 1\end{aligned}$$

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$$\begin{aligned}\Phi_{105}(q) = {} & q^{48} + q^{47} + q^{46} - q^{43} - q^{42} - 2q^{41} - q^{40} - q^{39} \\ & + q^{36} + q^{35} + q^{34} + q^{33} + q^{32} + q^{31} - q^{28} - q^{26} - q^{24} \\ & - q^{22} - q^{20} + q^{17} + q^{16} + q^{15} + q^{14} + q^{13} + q^{12} - q^{9} \\ & - q^{8} - 2q^{7} - q^{6} - q^{5} + q^{2} + q + 1\end{aligned}$$

q-binomials: factored and expanded

LEM factored

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \prod_{d=2}^{n} \Phi_d(q)^{\lfloor n/d \rfloor - \lfloor k/d \rfloor - \lfloor (n-k)/d \rfloor}$$

$\lfloor n/d \rfloor - \lfloor k/d \rfloor - \lfloor (n-k)/d \rfloor \in \{0,1\}$

proof

$$[n]_q! = \prod_{m=1}^{n} \prod_{\substack{d|m \\ d>1}} \Phi_d(q) = \prod_{d=2}^{n} \Phi_d(q)^{\lfloor n/d \rfloor}$$

□

- In particular, the q-binomial is a polynomial. (of degree $k(n-k)$)

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- In particular, the q-binomial is a polynomial. (of degree $k(n-k)$)

EG expanded

$$\binom{6}{2}_q = q^8 + q^7 + 2q^6 + 2q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1$$

$$\binom{9}{3}_q = q^{18} + q^{17} + 2q^{16} + 3q^{15} + 4q^{14} + 5q^{13} + 7q^{12} + 7q^{11} + 8q^{10} + 8q^9 + 8q^8 + 7q^7 + 7q^6 + 5q^5 + 4q^4 + 3q^3 + 2q^2 + q + 1$$

- The coefficients are positive and **unimodal**. Sylvester, 1878

q-binomials: Pascal's triangle

D2

THM The q-binomial satisfies the **q-Pascal rule**:

$$\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q$$

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$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 1+q & & 1 & \\ 1 & & 1+q(1+q) & & (1+q)+q^2 & & 1 \\ & & & \vdots & & & \end{array}$$

EG

$$\begin{aligned} \binom{4}{2}_q &= \binom{3}{1}_q + q^2 \binom{3}{2}_q \\ &= (1+q+q^2) + q^2(1+q+q^2) = 1+q+2q^2+q^3+q^4 \end{aligned}$$

q-binomials: combinatorial

D3

THM

$$\binom{n}{k}_q = \sum_Y q^{w(Y)} \quad \text{where } w(Y) = \overbrace{\sum_j y_j - j}^{\text{“normalized sum of } Y\text{”}}$$

The sum is over all k-element subsets Y of $\{1, 2, \ldots, n\}$.

EG

$$\underbrace{\{1,2\}}_{\to 0},\quad \underbrace{\{1,3\}}_{\to 1},\quad \underbrace{\{1,4\}}_{\to 2},\quad \underbrace{\{2,3\}}_{\to 2},\quad \underbrace{\{2,4\}}_{\to 3},\quad \underbrace{\{3,4\}}_{\to 4}$$

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$$\binom{4}{2}_q = 1 + q + 2q^2 + q^3 + q^4$$

The coefficient of q^m in $\binom{n}{k}_q$ counts the number of

- k-element subsets of n whose normalized sum is m,

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“normalized sum of Y”

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The coefficient of q^m in $\binom{n}{k}_q$ counts the number of

- k-element subsets of n whose normalized sum is m,
- partitions λ of m whose Ferrer’s diagram fits in a $k \times (n-k)$ box.

q-binomials: algebraic

THM Let q be a prime power.

$$\binom{n}{k}_q = \text{number of } k\text{-dim. subspaces of } \mathbb{F}_q^n$$

D4

proof

$\square$

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proof

- Number of ways to choose k linearly independent vectors in $\mathbb{F}_q^n$:

$$(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})$$

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- Number of ways to choose k linearly independent vectors in $\mathbb{F}_q^n$:

$$(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})$$

- Hence the number of k-dim. subspaces of $\mathbb{F}_q^n$ is:

$$\frac{(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})}{(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1})} = \binom{n}{k}_q \qquad \square$$

q-binomials: noncommuting variables

THM Suppose $yx = qxy$ (and that q commutes with x, y). Then:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j}_q x^j y^{n-j}$$

D5

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EG Part of the expansion of $(x+y)^4$:

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- Let $X \cdot f(x) = xf(x)$ and $Q \cdot f(x) = f(qx)$. Then:

$$QX \cdot f(x) = qxf(qx) = qXQ \cdot f(x)$$

DEF The **q-derivative**:

$$D_q f(x) = \frac{f(qx) - f(x)}{qx - x}$$

EG

$$D_q x^n$$

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$$D_q x^n = \frac{(qx)^n - x^n}{qx - x} = \frac{q^n - 1}{q - 1} x^{n-1} = [n]_q \, x^{n-1}$$

- $D_q e_q^x = e_q^x$
- $e_q^x \cdot e_q^y = e_q^{x+y}$
  provided that $yx = qxy$
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$$\int_0^x f(x) \, \mathrm{d}_q x := (1-q) \sum_{n=0}^{\infty} q^n x f(q^n x)$$

- The q-**gamma function**:

$$\Gamma_q(s) = \int_0^{\infty} x^{s-1} e_{1/q}^{-qx} \, \mathrm{d}_q x$$

D6

Can similarly define q-**beta** via a q-Euler integral.

- $\Gamma_q(s+1) = [s]_q \, \Gamma_q(s)$
- $\Gamma_q(n+1) = [n]_q!$

Summary: the q-binomial coefficient

The q-binomial coefficient has a variety of natural characterizations:

- $\binom{n}{k}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!}$
- **Recursively**, via a q-version of Pascal's rule
- **Combinatorially**, as a weighted counting of k-subsets of an n-set
- **Geometrically**, as the number of k-dimensional subspaces of $\mathbb{F}_q^n$
- **Algebraically**, via a binomial theorem for noncommuting variables
- **Analytically**, via q-integral representations

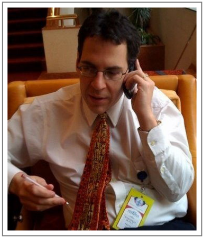
Binomial coefficients with integer entries

$$\binom{-3}{5} = -21, \qquad \binom{-3}{-5} = 6$$

$$\binom{-3.001}{-5.001} \approx 6.004$$

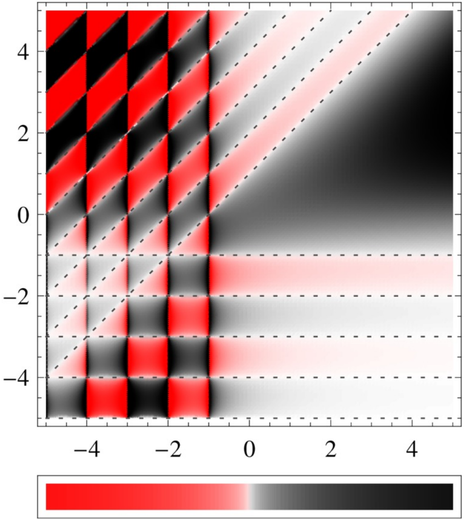
$$\binom{-3.003}{-5.005} \approx 10.03$$

Daniel E. Loeb
Sets with a negative number of elements
Advances in Mathematics, Vol. 91, p.64–74, 1992



1989: Ph.D. at MIT (Rota)
1996+: in mathematical finance

A function in two variables



This scale is also visible along the line $y = 1$.

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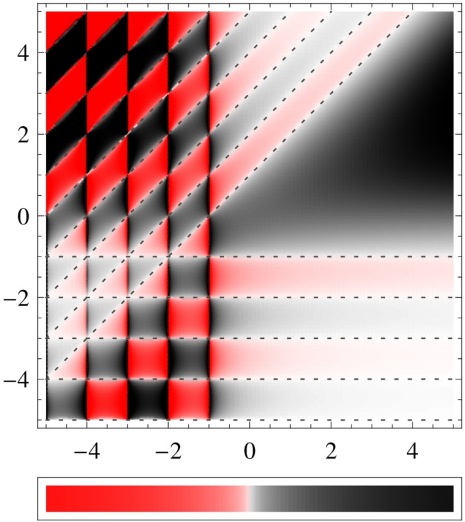

This scale is also visible along the line $y = 1$.

This is a plot of:

$$\binom{x}{y} := \frac{\Gamma(x+1)}{\Gamma(y+1)\Gamma(x-y+1)}$$

Defined and smooth on $\mathbb{R}^2 \backslash \{x = -1, -2, \ldots\}$.

“ ... no evidence that the graph of C has ever been plotted before ... ”
David Fowler, *American Mathematical Monthly, Jan 1996*

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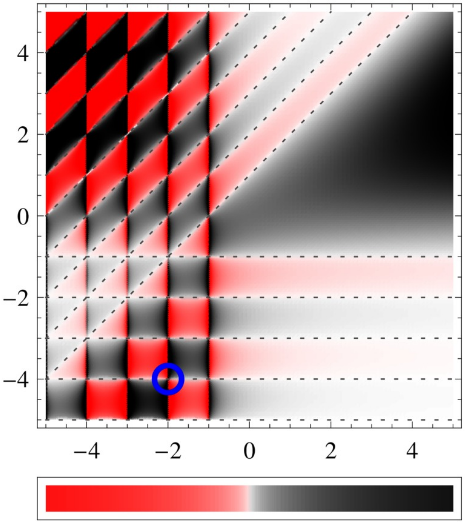

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Directional limits exist at integer points:

$$\lim_{\varepsilon \to 0} \binom{-2+\varepsilon}{-4+r\varepsilon} = \frac{1}{2!} \lim_{\varepsilon \to 0} \frac{\Gamma(-1+\varepsilon)}{\Gamma(-3+r\varepsilon)} = 3r$$

since $\Gamma(-n+\varepsilon) = \frac{(-1)^n}{n!} \frac{1}{\varepsilon} + O(1)$

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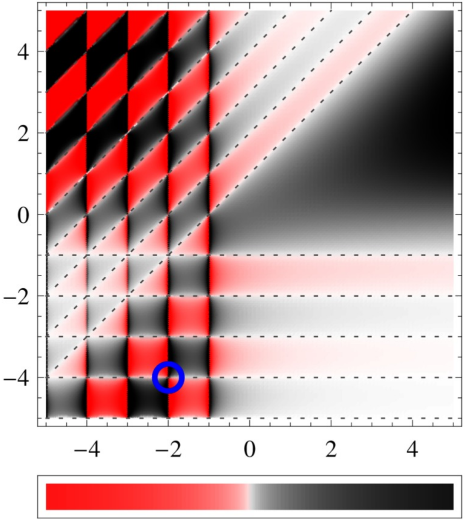

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DEF For all $x, y \in \mathbb{Z}$:

$$\binom{x}{y} := \lim_{\varepsilon \to 0} \frac{\Gamma(x+1+\varepsilon)}{\Gamma(y+1+\varepsilon)\Gamma(x-y+1+\varepsilon)}$$

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DEF **Hybrid sets and their subsets**

negative multiplicity

$$\{\, 1, 1, 4 \mid 2, 3, 3 \,\}$$

positive multiplicity

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negative multiplicity

$$\{\,1,1,4\;|\;2,3,3\,\}$$

positive multiplicity

$Y \subset X$ if one can repeatedly **remove** elements from X and thus obtain Y or have removed Y.

removing = decreasing the multiplicity of an element with nonzero multiplicity

EG Subsets of $\{1,1,4|2,3,3\}$ include:

(remove 4) $\{4|\}$, $\{1,1|2,3,3\}$

Sets with a negative number of elements

DEF **Hybrid sets and their subsets**

$$\{\underbrace{1,1,4}_{\text{positive multiplicity}} \mid \overbrace{2,3,3}^{\text{negative multiplicity}}\}$$

$\boldsymbol{Y} \subset \boldsymbol{X}$ if one can repeatedly **remove** elements from X and thus obtain Y or have removed Y.

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EG Subsets of $\{1,1,4|2,3,3\}$ include:

(remove 4)	$\{4\|\}$,	$\{1,1\|2,3,3\}$
(remove $4,2,2$)	$\{2,2,4\|\}$,	$\{1,1\|2,2,2,3,3\}$

Note that we cannot remove 4 again. $\{4,4|\}$ is not a subset.

Counting subsets of hybrid sets

- **New sets:**

$$\underset{\text{all multiplicities } 0,\,1}{\overset{3 \text{ elements:}}{\{1,2,4|\}}} \quad \text{or} \quad \underset{\text{all multiplicities } 0,\,-1}{\overset{-4 \text{ elements:}}{\{|1,2,4,5\}}}$$

THM Loeb 1992: For all integers n and k, the number of k-element subsets of an n-element new set is $\left|\binom{n}{k}\right|$.

Counting subsets of hybrid sets

- **New sets:** 3 elements: $\{1, 2, 4|\}$ (all multiplicities $0, 1$) or -4 elements: $\{|1, 2, 4, 5\}$ (all multiplicities $0, -1$)

THM (Loeb 1992) For all integers n and k, the number of k-element subsets of an n-element new set is $\left|\binom{n}{k}\right|$.

EG A usual set like $\{1, 2, 3|\}$ only has the usual subsets.

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EG A usual set like $\{1,2,3|\}$ only has the usual subsets.

EG $n=-3$

- $\left|\binom{-3}{2}\right| = 6$ because the 2-element subsets of $\{|1,2,3\}$ are:

$$\{1,1|\}, \quad \{1,2|\}, \quad \{1,3|\}, \quad \{2,2|\}, \quad \{2,3|\}, \quad \{3,3|\}$$

- **New sets:**

3 elements: $\{1,2,4|\}$ all multiplicities $0,1$ or -4 elements: $\{|1,2,4,5\}$ all multiplicities $0,-1$

THM (Loeb 1992) For all integers n and k, the number of k-element subsets of an n-element new set is $\left|\binom{n}{k}\right|$.

EG A usual set like $\{1,2,3|\}$ only has the usual subsets.

EG ($n=-3$)

- $\left|\binom{-3}{2}\right| = 6$ because the 2-element subsets of $\{|1,2,3\}$ are:

$$\{1,1|\}, \quad \{1,2|\}, \quad \{1,3|\}, \quad \{2,2|\}, \quad \{2,3|\}, \quad \{3,3|\}$$

- $\left|\binom{-3}{-4}\right| = 3$ because the -4-element subsets of $\{|1,2,3\}$ are:

$$\{|1,1,2,3\}, \quad \{|1,2,2,3\}, \quad \{|1,2,3,3\}$$

The binomial theorem

THM Loeb 1992

For all integers n and k, $$\binom{n}{k} = \{x^k\}(1+x)^n.$$

Here, we extract appropriate coefficients:

$$\{x^k\}f(x) := \begin{cases} a_k & \text{if } k \geqslant 0 \\ b_k & \text{if } k < 0 \end{cases}$$

around $x = 0$:

$$f(x) = \sum_{k \geqslant k_0} a_k x^k$$

around $x = \infty$:

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EG

$$(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + 15x^4 + O(x^5) \quad \text{as } x \to 0$$

$$(1+x)^{-3} = x^{-3} - 3x^{-4} + 6x^{-5} + O(x^{-6}) \quad \text{as } x \to \infty$$

Hence, for instance,

$$\binom{-3}{4} = 15, \qquad \binom{-3}{-5} = 6.$$

q-binomial coefficients with integer entries

DEF For all integers n and k,

$$\binom{n}{k}_q := \lim_{a\to q} \frac{(a;q)_n}{(a;q)_k (a;q)_{n-k}}.$$

$$\binom{-3}{4}_q = \frac{1}{q^{18}}(1-q+q^2)(1+q+q^2)(1+q+q^2+q^3+q^4)$$

$$\binom{-3}{-5}_q = \frac{1}{q^7}(1+q^2)(1+q+q^2)$$

S. Formichella, A. Straub
Gaussian binomial coefficients with negative arguments
Annals of Combinatorics, Vol. 23, Nr. 3, 2019, p. 725-748

The q-binomial theorem

THM Formichella S 2019: Suppose $yx = qxy$. For $n, k \in \mathbb{Z}$,

$$\binom{n}{k}_q = \{x^k y^{n-k}\}(x+y)^n.$$

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EG

$$(x+y)^{-1}$$

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EG

$$(x+y)^{-1} = ((xy^{-1}+1)y)^{-1}$$

$$\binom{-1}{k}_q$$

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EG

$$(x+y)^{-1} = ((xy^{-1}+1)y)^{-1} = y^{-1}(xy^{-1}+1)^{-1}$$
$$= y^{-1} \sum_{k \geqslant 0} (-1)^k (xy^{-1})^k$$

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EG

$$\begin{aligned}(x+y)^{-1} &= ((xy^{-1}+1)y)^{-1} = y^{-1}(xy^{-1}+1)^{-1} \\ &= y^{-1} \sum_{k \geqslant 0} (-1)^k (xy^{-1})^k \\ &= \sum_{k \geqslant 0} (-1)^k q^{-k(k+1)/2} x^k y^{-k-1}\end{aligned}$$

$$\binom{-1}{k}_q$$

Counting subsets of hybrid sets, q-version

THM Formichella S 2019

For all $n, k \in \mathbb{Z}$, $\binom{n}{k}_q = \varepsilon \sum_Y q^{\sigma(Y) - k(k-1)/2}$, $\varepsilon = \pm 1$.

The sum is over all k-element subsets Y of the n-element set X_n.

$\varepsilon = 1$ if $0 \leqslant k \leqslant n$. $\varepsilon = (-1)^k$ if $n < 0 \leqslant k$. $\varepsilon = (-1)^{n-k}$ if $k \leqslant n < 0$.

$$X_n := \begin{cases} \{0, 1, \ldots, n-1|\} & \text{if } n \geqslant 0 \\ \{|-1, -2, \ldots, n\} & \text{if } n < 0 \end{cases}$$

$$\sigma(Y) := \sum_{y \in Y} M_Y(y) y$$

$M_Y(y)$ is the multiplicity of y in Y.

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$$\sigma(Y) := \sum_{y \in Y} M_Y(y) y$$

$M_Y(y)$ is the multiplicity of y in Y.

EG $n = -3$

The -4-element subsets of $X_{-3} = \{| -1, -2, -3\}$ are:

$\{| -1, -1, -2, -3\}$, $\sigma = 7$ $\quad$ $\{| -1, -2, -2, -3\}$, $\sigma = 8$ $\quad$ $\{| -1, -2, -3, -3\}$, $\sigma = 9$

Hence, $\binom{-3}{-4}_q = -(q^{-3} + q^{-2} + q^{-1})$. (subtract $\frac{k(k-1)}{2} = 10$)

Conventions for binomial coefficients

Option advertised here:

$$\binom{n}{k} := \lim_{\varepsilon \to 0} \frac{\Gamma(n+1+\varepsilon)}{\Gamma(k+1+\varepsilon)\Gamma(n-k+1+\varepsilon)}$$

Alternative:

$$\binom{n}{k} := 0 \qquad \text{if } k < 0$$

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- used in Mathematica (at least 9+)
- used in Maple (at least 18+)
- used in SageMath (at least 8.0+)

EG MMA 12

```
Binomial[-3, -5]
> 6
```

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EG **MMA 12**

```
Binomial[-3, -5]
> 6
QBinomial[-3, -5, q]
> 0
```

Similarly, `expand(QBinomial(n,k,q))` in Maple 18 results in a division-by-zero error.

Application: Lucas congruences

THM Lucas 1878

Let p be prime. For integers $n, k \geqslant 0$,

$$\binom{n}{k} \equiv \binom{n_0}{k_0}\binom{n_1}{k_1}\binom{n_2}{k_2}\cdots \pmod{p},$$

where n_i, respectively k_i, are the p-adic digits of n and k.

EG

LHS $= 75{,}582$

$$\binom{19}{11} \equiv \binom{5}{4}\binom{2}{1} = 5 \cdot 2 \equiv 3 \pmod{7}$$

Application: Lucas congruences

THM Lucas 1878, Formichella S 2019

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LHS $= 75{,}582$

EG

$$\binom{-11}{-19} \equiv \binom{3}{2}\binom{5}{4}\binom{6}{6}\binom{6}{6}\cdots = 3 \cdot 5 \equiv 1 \pmod{7}$$

LHS $= 43{,}758$

Note the (infinite) 7-adic expansions:

$$-11 = 3 + 5 \cdot 7 + 6 \cdot 7^2 + 6 \cdot 7^3 + \ldots$$
$$-19 = 2 + 4 \cdot 7 + 6 \cdot 7^2 + 6 \cdot 7^3 + \ldots$$

Application: q-Lucas congruences

THM Olive 1965, Désarménien 1982

Let $m \geqslant 2$ be an integer. For integers $n, k \geqslant 0$,

$$\binom{n}{k}_q \equiv \binom{n_0}{k_0}_q \binom{n'}{k'} \quad (\operatorname{mod} \Phi_m(q)),$$

where $\begin{matrix} n = n_0 + n'm \\ k = k_0 + k'm \end{matrix}$ with $n_0, k_0 \in \{0, 1, \ldots, m-1\}$.

B. Adamczewski, J. P. Bell, and E. Delaygue.
Algebraic independence of G-functions and congruences "à la Lucas"
Annales Scientifiques de l'École Normale Supérieure, 2016

Application: q-Lucas congruences

THM Olive 1965, Désarménien 1982, Formichella S 2019

Let $m \geqslant 2$ be an integer. For **all integers** n, k,

$$\binom{n}{k}_q \equiv \binom{n_0}{k_0}_q \binom{n'}{k'} \pmod{\Phi_m(q)},$$

where $\begin{array}{l} n = n_0 + n'm \\ k = k_0 + k'm \end{array}$ with $n_0, k_0 \in \{0, 1, \ldots, m-1\}$.

EG

$$\binom{-11}{-19}_q \equiv \binom{3}{2}_q \binom{-2}{-3} = -2(1+q+q^2) \pmod{\Phi_7(q)}$$

- LHS $= \frac{1}{q^{116}}(1 + q + 2q^2 + 3q^3 + 5q^4 + \ldots + q^{80})$
- $q = 1$ reduces to $\binom{-11}{-19} \equiv -6 \equiv 1 \pmod 7$.

B. Adamczewski, J. P. Bell, and E. Delaygue.
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Advertisement: More Lucas congruences

Apéry's proof of the irrationality of $\zeta(3)$ centers around:

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

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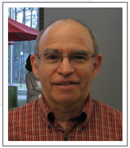
THM Gessel 1982

$$A(n) \equiv A(n_0)A(n_1) \cdots A(n_r) \pmod{p},$$

where n_i are the p-adic digits of n.



- Gessel's approach generalized by McIntosh (1992)

R. J. McIntosh
A generalization of a congruential property of Lucas.
Amer. Math. Monthly, Vol. 99, Nr. 3, 1992, p. 231–238

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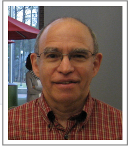


- Gessel's approach generalized by McIntosh (1992)
- $6+6+3$ sporadic Apéry-like sequences are known.

THM Malik–S 2015

Every (known) sporadic sequence satisfies these Lucas congruences modulo every prime.

A. Malik, A. Straub
Divisibility properties of sporadic Apéry-like numbers
Research in Number Theory, Vol. 2, Nr. 1, 2016, p. 1–26

R. J. McIntosh
A generalization of a congruential property of Lucas.
Amer. Math. Monthly, Vol. 99, Nr. 3, 1992, p. 231–238

Application: Apéry number supercongruences

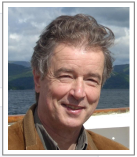
The **Apéry numbers**

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy many interesting properties, including **supercongruences**: $p \geqslant 5$ prime

THM Beukers 1985

$$A(p^r m - 1) \equiv A(p^{r-1} m - 1) \pmod{p^{3r}}$$



THM Coster 1988

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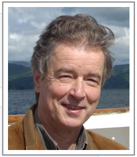

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$$A(p^r m)\equiv A(p^{r-1}m) \pmod{p^{3r}}$$

- Extend $A(n)$ to integers n:

$$A(n)=\sum_{k\in\mathbb{Z}}\binom{n}{k}^2\binom{n+k}{k}^2$$

- It then follows that:

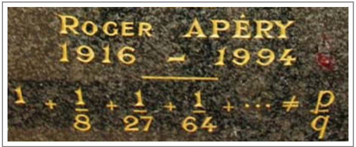
$$A(-n)=A(n-1)$$

Uniform proof (and explanation) of Beukers/Coster supercongruences

Apéry numbers

CONJ $\pi, \zeta(3), \zeta(5), \ldots$ are algebraically independent over $\mathbb{Q}$.

- Apéry (1978): $\zeta(3)$ is irrational
- Open: $\zeta(5)$ is irrational
- Open: $\zeta(3)$ is transcendental
- Open: $\zeta(3)/\pi^3$ is irrational
- Open: Catalan's constant $G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$ is irrational



A. Straub
Supercongruences for polynomial analogs of the Apéry numbers
Proceedings of the American Mathematical Society, Vol. 147, 2019, p. 1023-1036

Apéry numbers and the irrationality of $\zeta(3)$

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2+17n+5)u_n - n^3 u_{n-1}.$$

THM Apéry '78: $\zeta(3) = \sum_{n \geqslant 1} \frac{1}{n^3}$ is irrational.

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proof The same recurrence is satisfied by the "near"-integers

$$B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{j=1}^{n} \frac{1}{j^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right).$$

Then, $\frac{B(n)}{A(n)} \to \zeta(3)$. But too fast for $\zeta(3)$ to be rational. □

Zagier's search and Apéry-like numbers

- The Apéry numbers $B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}$ for $\zeta(2)$ satisfy

$$(n+1)^2 u_{n+1} = (an^2 + an + b)u_n - cn^2 u_{n-1}, \qquad (a,b,c) = (11,3,-1).$$

Q **Beukers** Are there other tuples (a,b,c) for which the solution defined by $u_{-1} = 0$, $u_0 = 1$ is integral?

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- Apart from degenerate cases, Zagier found 6 sporadic integer solutions:

A	$\sum_{k=0}^{n} \binom{n}{k}^3$	**D**	$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{n}$
B	$\sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k 3^{n-3k} \binom{n}{3k} \frac{(3k)!}{k!^3}$	**E**	$\sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} \binom{2(n-k)}{n-k}$
C	$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}$	**F**	$\sum_{k=0}^{n} (-1)^k 8^{n-k} \binom{n}{k} C_{\boldsymbol{A}}(k)$

Modularity of Apéry-like numbers

- The Apéry numbers $1, 5, 73, 1145, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$\underbrace{\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}}_{\text{modular form}} = \sum_{n \geqslant 0} A(n) \underbrace{\left(\frac{\eta^{12}(\tau)\eta^{12}(6\tau)}{\eta^{12}(2\tau)\eta^{12}(3\tau)} \right)^n}_{\text{modular function}}.$$

$1 + 5q + 13q^2 + 23q^3 + O(q^4)$ $\qquad$ $q - 12q^2 + 66q^3 + O(q^4)$

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FACT Not at all evidently, such a **modular parametrization** exists for all known Apéry-like numbers!

- Context:

$f(\tau)$	modular form of weight k
$x(\tau)$	modular function
$y(x)$	such that $y(x(\tau)) = f(\tau)$

Then $y(x)$ satisfies a linear differential equation of order $k+1$.

Supercongruences for Apéry numbers

- Chowla, Cowles and Cowles (1980) conjectured that, for $p \geqslant 5$,

$$A(p) \equiv 5 \pmod{p^3}.$$

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THM Beukers, Coster '85, '88: The Apéry numbers satisfy the **supercongruence** $(p \geqslant 5)$

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$$\binom{ap}{bp} \equiv \binom{a}{b} \pmod{p^3}$$
Ljunggren '52

- Conjecturally, supercongruences like

$$A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{3r}}$$

hold for all Apéry-like numbers. Osburn–Sahu '09

- Current state of affairs for the six sporadic sequences related to $\zeta(3)$:

$A(n)$	
$\sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$	Beukers, Coster '87-'88
$\sum_k \binom{n}{k}^2 \binom{2k}{n}^2$	Osburn–Sahu–S '16
$\sum_k \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$	Osburn–Sahu '11
$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$	open (modulo p^3 Amdeberhan–Tauraso '16)
$\sum_k (-1)^k \binom{n}{k}^3 \binom{4n-5k}{3n}$	Osburn–Sahu–S '16
$\sum_{k,l} \binom{n}{k}^2 \binom{n}{l} \binom{k}{l} \binom{k+l}{n}$	Gorodetsky '18

Non-super congruences are abundant

$$a(mp^r) \equiv a(mp^{r-1}) \pmod{p^r} \qquad \text{(G)}$$

- **realizable** sequences $a(n)$, i.e., for some map $T : X \to X$,

$$a(n) = \#\{x \in X : T^n x = x\} \qquad \text{“points of period } n\text{”}$$

Everest–van der Poorten–Puri–Ward ’02, Arias de Reyna ’05

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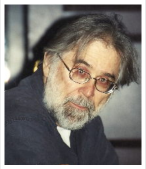
- (G) is equivalent to $\exp\left(\sum_{n=1}^{\infty} \frac{a(n)}{n} T^n\right) \in \mathbb{Z}[[T]]$.

  This is a natural condition in **formal group theory**.

q-congruences of Clark and Andrews



THM Clark 1995

$$\binom{an}{bn}_q \equiv \binom{a}{b}_{q^{n^2}} \pmod{\Phi_n(q)^2}$$

proof $a=2$, $b=1$

Combinatorially, we have q-Chu-Vandermonde:

$$\binom{2n}{n}_q = \sum_{k=0}^{n} \binom{n}{k}_q \binom{n}{n-k}_q q^{(n-k)^2}$$

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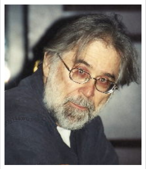

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(Note that $\Phi_n(q)$ divides $\binom{n}{k}_q$ unless $k=0$ or $k=n$.) □

- $\Phi_n(1) = 1$ if n is not a prime power.

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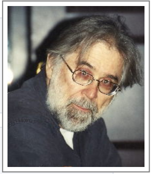

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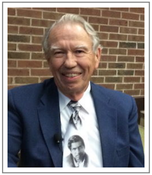
- $\Phi_n(1) = 1$ if n is not a prime power.
- Similar results by Andrews (1999); e.g.:

$$\binom{ap}{bp}_q \equiv q^{(a-b)b\binom{p}{2}} \binom{a}{b}_{q^p} \pmod{[p]_q^2}$$



A q-analog of Ljunggren's congruence

- The following answers Andrews' question to find a q-analog of Wolstenholme's congruence.

THM S 2011/18

$$\binom{an}{bn}_q \equiv \binom{a}{b}_{q^{n^2}} - b(a-b)\binom{a}{b}\frac{n^2-1}{24}(q^n-1)^2 \pmod{\Phi_n(q)^3}$$

EG $n = 13$, $a = 2$, $b = 1$

$$\binom{26}{13}_q = \underbrace{1+q^{169}}_{\to 2} - \underbrace{14(q^{13}-1)^2}_{\to 0} + \underbrace{(1+q+\ldots+q^{12})^3}_{\to 13^3} f(q)$$

where $f(q) = 14 - 41q + 41q^2 - \ldots + q^{132} \in \mathbb{Z}[q]$.

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- Note that $\frac{n^2-1}{24}$ is an integer if $(n,6) = 1$.

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- Note that $\frac{n^2-1}{24}$ is an integer if $(n,6)=1$.
- $\binom{ap}{bp} \equiv \binom{a}{b}$ holds modulo p^{3+r} where r is the p-adic valuation of

$$a\,b(a-b)\binom{a}{b}.$$

Jacobsthal 1952

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THM Zudilin 2019

Extension of above congruence to q-analog of ($p \geqslant 5$)

$$\binom{ap}{bp} \equiv \binom{a}{b} + ab(a-b)p\sum_{k=1}^{p-1}\frac{1}{k} \pmod{p^4}.$$

Q Creative microscoping à la Guo and Zudilin?
Extra parameter c and congruences modulo, say, $\Phi_n(q)(1-cq^n)(c-q^n)$.

A q-version of the Apéry numbers

- A symmetric q-analog of the Apéry numbers:

$$A_q(n) = \sum_{k=0}^{n} q^{(n-k)^2} \binom{n}{k}_q^2 \binom{n+k}{k}_q^2$$

This is an explicit form of a q-analog of Krattenthaler, Rivoal and Zudilin (2006).

EG The first few values are:

$$\begin{aligned} A(0) &= 1 & A_q(0) &= 1 \\ A(1) &= 5 & A_q(1) &= 1 + 3q + q^2 \\ A(2) &= 73 & A_q(2) &= 1 + 3q + 9q^2 + 14q^3 + 19q^4 + 14q^5 \\ & & &\quad + 9q^6 + 3q^7 + q^8 \\ A(3) &= 1445 & A_q(3) &= 1 + 3q + 9q^2 + 22q^3 + 43q^4 + 76q^5 \\ & & &\quad + 117q^6 + \ldots + 3q^{17} + q^{18} \end{aligned}$$

THM
S
2014/18

The q-analog of the Apéry numbers, defined as

$$A_q(n) = \sum_{k=0}^{n} q^{(n-k)^2} \binom{n}{k}_q^2 \binom{n+k}{k}_q^2,$$

satisfies, for any $m \geqslant 0$,

$A_q(1) = 1 + 3q + q^2, \quad A(1) = 5$

$$A_q(mn) \equiv A_{q^{m^2}}(n) - \frac{m^2-1}{12}(q^m-1)^2 n^2 A_1(n) \pmod{\Phi_m(q)^3}.$$

THM
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2014/18

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- Gorodetsky (2018) recently proved q-congruences implying the stronger congruences $A(p^r n) \equiv A(p^{r-1} n)$ modulo p^{3r}.

Q

q-analog and congruences for Almkvist–Zudilin numbers?

$$\sum_k (-3)^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$$

(classical supercongruences still open)

Multivariate supercongruences

Q q-analog and congruences for Almkvist–Zudilin numbers?

$$Z(n) = \sum_k (-3)^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$$

(classical supercongruences still open)

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EG S 2014 The Almkvist–Zudilin numbers are the diagonal Taylor coefficients of

$$\frac{1}{1-(x_1+x_2+x_3+x_4)+27x_1x_2x_3x_4} = \sum_{\boldsymbol{n}\in\mathbb{Z}_{\geqslant 0}^4} Z(\boldsymbol{n})\boldsymbol{x}^{\boldsymbol{n}}$$

CONJ S 2014 For $p \geqslant 5$, we have the multivariate supercongruences

$$Z(\boldsymbol{n}p^r) \equiv Z(\boldsymbol{n}p^{r-1}) \pmod{p^{3r}}.$$

THANK YOU!

Slides for this talk will be available from my website:
`http://arminstraub.com/talks`

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Gaussian binomial coefficients with negative arguments
Annals of Combinatorics, Vol. 23, Nr. 3, 2019, p. 725-748

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Proceedings of the American Mathematical Society, Vol. 147, 2019, p. 1023-1036