Midterm #1

MATH 481/581 — Cryptography
Monday, Feb 23, 2026

Please print your name:

No notes, calculators or tools of any kind are permitted. There are 40 points in total. You need to show work to receive full credit.

Good luck!

Problem 1. (7 points) Using the Chinese remainder theorem, determine all solutions to $x^2 \equiv 9 \pmod{55}$.

Solution. By the CRT:

$$\begin{aligned} & x^2 \equiv 9 \pmod{55} \\ \iff\ & x^2 \equiv 9 \pmod{5} \text{ and } x^2 \equiv 9 \pmod{11} \\ \iff\ & x \equiv \pm 3 \pmod{5} \text{ and } x \equiv \pm 3 \pmod{11} \end{aligned}$$

Hence, there are four solutions $\pm 3, \pm a$ modulo 55. To find one of the nontrivial ones, we solve the congruences $x \equiv 3 \pmod{5}$, $x \equiv -3 \pmod{11}$:

$$x \equiv 3 \cdot 11 \cdot \underbrace{11^{-1}_{\bmod 5}}_{1} - 3 \cdot 5 \cdot \underbrace{5^{-1}_{\bmod 11}}_{-2} \equiv 33 + 30 \equiv 8 \pmod{55}$$

Hence, we conclude that $x^2 \equiv 9 \pmod{55}$ has the four solutions $\pm 3, \pm 8 \pmod{55}$.

Problem 2. (4 points)

(a) Suppose N is composite. x is a Fermat liar modulo N if and only if [ ].

(b) $8 \pmod{21}$ ☐ is a Fermat liar ☐ is not a Fermat liar because [ ].

Solution.

(a) x is a Fermat liar modulo N if and only if $x^{N-1} \equiv 1 \pmod{N}$.

(b) 8 is a Fermat liar modulo 21 if and only if $8^{20} \equiv 1 \pmod{21}$.

$8^2 \equiv 1 \pmod{21}$, so that $8^{20} \equiv 1 \pmod{21}$. Hence, 8 a Fermat liar modulo 21.

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Problem 3. (2 points) Briefly outline the Fermat primality test.

Solution. Fermat primality test:

Input: number n and parameter k indicating the number of tests to run
Output: "not prime" or "possibly prime"
Algorithm:

Repeat k times:
  Pick a random number a from $\{2, 3, \ldots, n-2\}$.
  If $a^{n-1} \not\equiv 1 \pmod{n}$, then stop and output "not prime".
Output "possibly prime".

Problem 4. (6 points) Evaluate $23^{1613} \pmod{17}$. Show your work!

Solution. First, $26^{1613} \equiv 6^{1613} \pmod{17}$. Since $1613 \equiv 13 \pmod{\phi(17)}$, we have $6^{1613} \equiv 6^{13} \pmod{17}$.

Using binary exponentiation, we find $6^2 \equiv 2 \pmod{17}$, $6^4 \equiv 2^2 = 4 \pmod{17}$, $6^8 \equiv 4^2 \equiv -1 \pmod{17}$.

In conclusion, $26^{1613} \equiv 6^{13} = 6^8 \cdot 6^4 \cdot 6 \equiv -1 \cdot 4 \cdot 6 \equiv 10 \pmod{17}$.

Problem 5. (6 points) Eve intercepts the ciphertext $c = (001\ 001\ 001)_2$. She knows it was encrypted with a stream cipher using the linear congruential generator $x_{n+1} \equiv 5x_n + 3 \pmod{8}$ as PRG.

Eve further knows that the plaintext begins with $m = (011\ 1\ldots)_2$. Break the cipher and determine the plaintext.

Solution. Since $c = m \oplus \mathrm{PRG}$, we learn that the initial piece of the keystream is $\mathrm{PRG} = c \oplus m = (001\ 001\ 001)_2 \oplus (011\ 1\ldots)_2 = (010\ 1\ldots)_2$.

Since each x_n has 3 bits, we learn that $x_1 = (010)_2 = 2$. Using $x_{n+1} \equiv 5x_n + 3 \pmod{8}$, we find $x_2 = 5$, $x_3 = 4$, ... In other words, $\mathrm{PRG} = 2, 5, 4, \ldots = (010\ 101\ 100\ \ldots)_2$.

Hence, Eve can decrypt the ciphertext and obtain $m = c \oplus \mathrm{PRG} = (001\ 001\ 001)_2 \oplus (010\ 101\ 100)_2 = (011\ 100\ 101)_2$.

Problem 6. (15 points) Fill in the blanks.

(a) In order for a PRG to be suitable for use in a stream cipher, the PRG must be [ ].

(b) Recall that, in a stream cipher, we must never reuse the key stream.

Nevertheless, we can reuse the key if we use a [ ].

(c) Using a one-time pad and key $k=(0011)_2$, the message $m=(1010)_2$ is encrypted to [ ].

(d) While perfectly confidential, the one-time pad does not protect against [ ].

(e) The LFSR $x_{n+15} \equiv x_{n+14} + x_n \pmod 2$ must repeat after [ ] terms.

(f) The first 5 bits generated by the Blum-Blum-Shub PRG with $M=133$ using the seed 5 are [ ].

You may use that $16^2 \equiv 123$, $25^2 \equiv 93$, $36^2 \equiv 99$, $92^2 \equiv 85$, $93^2 \equiv 4$, $99^2 \equiv 92 \pmod{133}$.

(g) Despite its flaws, in which scenario is it fine to use the Fermat primality test?

[ ]

(h) 30 in base 2 is [ ].

(i) The residue x is invertible modulo n if and only if [ ].

(j) $2^{-1} \pmod{13} \equiv$ [ ].

(k) Modulo 55, there are [ ] invertible residues, of which [ ] are quadratic.

(l) Modulo 31, there are [ ] invertible residues, of which [ ] are quadratic.

(m) We have $\phi(ab) = \phi(a)\phi(b)$ provided that [ ].

(n) How many solutions does the congruence $x^2 \equiv 4 \pmod{231}$ have? [ ] $231 = 3 \cdot 7 \cdot 11$

(o) How many solutions does the congruence $x^2 \equiv 36 \pmod{231}$ have? [ ]

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Solution.

(a) In order for a PRG to be suitable for use in a stream cipher, the PRG must be unpredictable.

(b) We can reuse the key if we use a nonce.

(c) Using a one-time pad and key $k = (0011)_2$, the message $m = (1010)_2$ is encrypted to $(1001)_2$.

(d) While perfectly confidential, the one-time pad does not protect against tampering.

(e) The LFSR $x_{n+15} \equiv x_{n+14} + x_n \pmod 2$ must repeat after $2^{15} - 1$ terms.

(f) The first five bits generated by the Blum-Blum-Shub PRG with $M = 133$ using the seed 5 are $1, 1, 0, 0, 1$ (obtained from $25, 93, 4, 16, 123$).

(g) Despite its flaws, it is fine to use the Fermat primality test for large random numbers.

(h) 30 in base 2 is $(11110)_2$.

(i) The residue x is invertible modulo n if and only if $\gcd(x, n) = 1$.

(j) $2^{-1} \pmod{13} \equiv 7$.

(k) Modulo 55, there are $\phi(55) = \phi(5)\phi(11) = 40$ invertible residues, of which $\frac{1}{4}\phi(55) = 10$ are quadratic.

(l) Modulo the prime 31, there are $\phi(31) = 30$ invertible residues, of which $\frac{1}{2}\phi(31) = 15$ are quadratic.

(m) We have $\phi(ab) = \phi(a)\phi(b)$ provided that $\gcd(a, b) = 1$.

(n) By the CRT, since $231 = 3 \cdot 7 \cdot 11$, the congruence $x^2 \equiv 4 \pmod{231}$ has $2 \cdot 2 \cdot 2 = 8$ solutions.

(o) $x^2 \equiv 36 \pmod{231}$ only has $1 \cdot 2 \cdot 2 = 4$ solutions because $x^2 \equiv 36 \pmod 3$ only has one solution (namely, $x \equiv 0$).

(extra scratch paper)