Midterm #1

MATH 316 — Linear Algebra II
Monday, Feb 23, 2026

Please print your name:

No notes, calculators or tools of any kind are permitted. There are 32 points in total. You need to show work to receive full credit.

Good luck!

Problem 1. (7 points) Determine the QR decomposition of the matrix $\begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 1 \end{bmatrix}$.

Solution. Let $\boldsymbol{w}_1, \boldsymbol{w}_2$ be the two columns of the matrix. We first construct an orthogonal basis $\boldsymbol{q}_1, \boldsymbol{q}_2$ using Gram–Schmidt (and then normalize afterwards):

- $\boldsymbol{q}_1 = \boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$
- $\boldsymbol{q}_2 = \boldsymbol{w}_2 - \frac{\boldsymbol{w}_2 \cdot \boldsymbol{q}_1}{\boldsymbol{q}_1 \cdot \boldsymbol{q}_1} \boldsymbol{q}_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - \frac{9}{9} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

Normalizing, we obtain the orthonormal basis $\frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. Hence, $Q = \begin{bmatrix} 1/3 & 0 \\ 2/3 & 1/\sqrt{2} \\ 2/3 & -1/\sqrt{2} \end{bmatrix}$.

We compute that $R = Q^T A = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 0 & \sqrt{2} \end{bmatrix}$.

Problem 2. (3 points) We want to find values for the parameters a, b, c such that $z = a + bx^2 + cy$ best fits some given points $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots$ Set up a linear system such that $[a, b, c]^T$ is a least squares solution.

Solution. The equations $a + bx_i^2 + cy_i = z_i$ translate into the system:

$$\underbrace{\begin{bmatrix} 1 & x_1^2 & y_1 \\ 1 & x_2^2 & y_2 \\ 1 & x_3^2 & y_3 \\ \vdots & \vdots & \vdots \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \underbrace{\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \end{bmatrix}}_{\boldsymbol{z}}$$

Of course, this is usually inconsistent. To find the best possible a, b, c we compute a least squares solution.

Armin Straub
straub@southalabama.edu

Problem 3. (6 points)

(a) Find the least squares solution to $\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$.

(b) Determine the least squares line for the data points $(2,-1),(1,2),(1,0)$.

Solution. Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$. Clearly, $A^T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$.

(a) We compute $A^T A = \begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix}$ and $A^T \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Hence, the normal equations $A^T A \hat{\boldsymbol{x}} = A^T \boldsymbol{b}$ are

$$\begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix} \hat{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Solving, we find that the least squares solution is $\hat{\boldsymbol{x}} = \frac{1}{2}\begin{bmatrix} 6 & -4 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

(b) We need to determine the values a, b for the least squares line $y = a + bx$. The equations $a + bx_i = y_i$ translate into the system

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \quad \text{that is,} \quad \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}.$$

We have already computed that the least squares solution to that system is $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

Hence, the least squares line is $y = 3 - 2x$.

Problem 4. (4 points) Diagonalize the symmetric matrix $A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$ as $A = PDP^T$.

Solution. The characteristic polynomial is $\begin{vmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = (2-\lambda)(-1-\lambda) - 4 = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2)$, and so A has eigenvalues $3, -2$.

The 3-eigenspace is $\operatorname{null}\left(\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Normalized: $\frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

The -2-eigenspace is $\operatorname{null}\left(\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Normalized: $\frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Hence, if $P = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$, then $A = PDP^T$.

Armin Straub
straub@southalabama.edu

Problem 5. (4 points) Consider the vector space $W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(a) Determine the orthogonal projection of $\begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}$ onto W.

(b) Determine the orthogonal projection of that same vector onto $W^{\perp}$.

Solution.

(a) Note that $\boldsymbol{q}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\boldsymbol{q}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ are orthogonal. Hence, the orthogonal projection of $\boldsymbol{v} = \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}$ onto W is

$$\frac{\boldsymbol{v} \cdot \boldsymbol{q}_1}{\boldsymbol{q}_1 \cdot \boldsymbol{q}_1} \boldsymbol{q}_1 + \frac{\boldsymbol{v} \cdot \boldsymbol{q}_2}{\boldsymbol{q}_2 \cdot \boldsymbol{q}_2} \boldsymbol{q}_2 = \frac{6}{6} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \frac{-5}{5} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

(b) We can compute this as the error of the projection in the previous part: $\begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ -2 \end{bmatrix}$.

Problem 6. (3 points) Let $A = \begin{bmatrix} 1 & 3 & 0 & 2 & 4 \\ 0 & 0 & 1 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) A basis for null(A) is ☐. A basis for col(A) is ☐.

(b) $\dim \operatorname{col}(A) =$ ☐, $\dim \operatorname{row}(A) =$ ☐, $\dim \operatorname{null}(A) =$ ☐, $\dim \operatorname{null}(A^T) =$ ☐

Solution.

(a) A basis for null(A) is $\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 5 \\ 0 \\ 1 \end{bmatrix}$. A basis for col($A$) is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

(b) $\dim \operatorname{col}(A) = 2$, $\dim \operatorname{row}(A) = 2$, $\dim \operatorname{null}(A) = 3$, $\dim \operatorname{null}(A^T) = 1$

Problem 7. (5 points) Fill in the blanks.

(a) col(A) is the orthogonal complement of ☐. null(A) is the orthogonal complement of ☐.

(b) If A is a 4×8 matrix with rank 3, then $\dim \operatorname{null}(A) =$ ☐ and $\dim \operatorname{row}(A) =$ ☐.

(c) By definition, a matrix Q is orthogonal if and only if ☐.

(d) The linear system $A\boldsymbol{x} = \boldsymbol{b}$ is consistent if and only if $\boldsymbol{b}$ is orthogonal to $\boxed{}$.

(e) Let W be the subspace of $\mathbb{R}^6$ of all solutions to $x_1 - x_4 + 3x_6 = 0$. $\dim W = \boxed{}$ and $\dim W^{\perp} = \boxed{}$.

Solution.

(a) $\mathrm{col}(A)$ is the orthogonal complement of $\mathrm{null}(A^T)$. $\mathrm{null}(A)$ is the orthogonal complement of $\mathrm{col}(A^T)$.

(b) If A is a 4×8 matrix with rank 3, then $\dim \mathrm{null}(A) = 8 - 3 = 5$ and $\dim \mathrm{row}(A) = 3$.

(c) By definition, a matrix Q is orthogonal if and only if Q is $n \times n$ (square) and Q has orthonormal columns.

(d) The linear system $A\boldsymbol{x} = \boldsymbol{b}$ is consistent if and only if $\boldsymbol{b}$ is orthogonal to $\mathrm{null}(A^T)$.

(e) If W is the space of all solutions to $x_1 - x_4 + 3x_6 = 0$ (in other words, $W = \mathrm{null}([\, 1 \;\; 0 \;\; 0 \;\; -1 \;\; 0 \;\; 3 \,])$), then $\dim W = 5$ and $\dim W^{\perp} = 1$.

(extra scratch paper)